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Drag on a sphere in slow shear flow

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A general closed form for the mobility tensor of a sphere moving in a fluid in stationary homogeneous flow is derived using the induced force method up to the first order in the square root of the Reynolds number based on the velocity gradient of the unperturbed flow. The closed form for the mobility tensor is valid for the time-dependent case as well as for the stationary case. As a special case, we calculate it explicitly for a simple shear flow. The result for the x, z-component for the stationary case, which gives the lift force, agrees with the value calculated by Saffman.

1. Introduction

Saffman (1965) calculated the lift force acting on a sphere moving with a constant velocity relative to a fluid in shear flow. He showed that if a sphere with radius a moves with a relative velocity, u, along the direction of the unperturbed velocity field, it feels a force perpendicular to the direction of the velocity of the sphere and toward the streamlines moving in the direction opposite to u, which is given by

$$F_{lift} = 6\pi a\eta u \times 0.343 Re_\beta^{1/2}. \tag{1.1}$$

Here η is the shear viscosity and $Re_\beta \equiv \beta a^2/\nu$ is the Reynolds number defined in terms of the shear rate β, the radius and the kinematic viscosity $\nu = \eta/\rho$, where ρ is the density of the fluid. He derived (1.1) by solving the Navier–Stokes equation with stick boundary conditions using the method of matched asymptotic expansions. He assumed that the relative velocity of the sphere is small enough such that

$$Re_u \ll Re_\beta^{1/2} \ll 1, \tag{1.2}$$

where $Re_u \equiv ua/\nu$ is the Reynolds number based on the translational velocity of the sphere. He concluded that the lift force on the sphere has its origin in the disturbance of the velocity field at the distance where the viscous effect becomes comparable with the convective effect and where the sphere can be regarded as a point force. Saffman's result was extended straightforwardly to the case where the particle is moving with a constant velocity in an arbitrary direction relative to shear flow by Harper & Chang (1968).

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The method of matched asymptotic expansions was also employed in the calculation of the drag on the particle in pure rotational flow by Herron, Davis & Bretherton (1975), Drew (1978), and Gotoh (1990), and in elongational flow by Drew (1978).

Mclaughlin (1991) extended Saffman's result to the case where $Re_u, Re_\beta^{1/2} \ll 1$. He solved the Navier–Stokes equation with a point force located at the centre of the coordinate system. He showed that, as Re_u increases, the lift force decreases monotonically to a very small value. All the authors referred to above assumed that both Reynolds numbers are much smaller than unity.

The purpose of this paper is to give the mobility tensor up to the first order in the square root of Re_β for the time-dependent motion of the sphere as well as for the stationary case, assuming the same conditions as Saffman's, (1.2), for the Reynolds numbers involved. For this purpose, we use the induced force method. This method was introduced by Mazur & Bedeaux (1974) to generalize Faxén's theorem to the time-dependent case. The advantage of this method is that one does not have to evaluate the velocity field explicitly by solving the Navier–Stokes equation in order to calculate the drag force exerted on the sphere. This method has subsequently been used to calculate the mobility tensor in the case where the sphere is moving along the axis of a centrifuge (Weisenborn 1985), in pure elongational flow (Bedeaux & Rubi 1987) and in arbitrary homogeneous flow (Pérez-Madrid, Rubi & Bedeaux 1990). However, both Bedeaux *et al.* and Pérez-Madrid *et al.* used an approximate expression for the velocity field due to a point force. This resulted in a value of the numerical coefficient in the lift force which differs from the one found by Saffman. The approximation was made in the Fourier-transformed Navier–Stokes equation where a non-algebraic term was neglected which, on the basis of symmetry considerations, was argued to give contributions of a higher order in Re_β. Owing to the divergent nature of the contributions due to this term, this argument is not correct. Though the approximation also leads in a simple and straightforward manner to results which are qualitatively correct for the matrix elements which were not given by Saffman, the numerical values of the coefficients differ from the exact ones. In this paper, we address this problem and show that the induced force method reproduces the lift force given by Saffman if one uses the exact expression for the velocity field due to a point force. For this purpose, we use a method developed by Onuki & Kawasaki (1980), where the exact solution is constructed by the introduction of a time-dependent wavevector. Saffman's result is generalized by taking the effect of the acceleration of the sphere into account and calculating the full mobility tensor up to the same order in Re_β. A closed form for the mobility, valid for non-stationary motion in arbitrary homogeneous flow, is given in terms of a Green function. Using this closed form, we analyse the large-frequency regime and the stationary case in detail.

This paper is organized as follows. In §2, we give the equations of motion which describe the dynamics of the fluid and of the sphere. Then the problem is reformulated by introducing the induced force field. In §3, we construct the general solution for the velocity field in terms of the induced force field through the Green function, which is by definition equal to the velocity field due to a point force, in terms of the time-dependent wavevector. Section 4 is devoted to the derivation of a closed form of the mobility tensor in terms of the Green function. We assume that the inertial effect of the fluid and the sphere is small on the time scale under the consideration. If one assumes that the mass density of the sphere is of the same order as that of the surrounding fluid, the time for the inertial effect to relax, τ_{in}, is approximately a^2/ν, while the time regime where the effect of the homogeneous flow becomes predominant is of order of $\tau_\beta = 1/\beta$. Since $\tau_{in}/\tau_\beta = Re_\beta \ll 1$, we focus on the time regime much larger than τ_{in}. This introduces another small dimensionless parameter, the radius multiplied by the inverse penetration depth, which is defined in terms of the frequency ω by $\alpha a \equiv (-\mathrm{i}\omega a^2/\nu)^{1/2}$ (with $\mathrm{Re}[\alpha] \geqslant 0$, where Re denotes the real part). The mobility tensor we derive is valid up to the first order in this parameter. As a special case, the mobility tensor for the simple shear case is evaluated in §5. There we focus on the two extreme cases. On the one hand, we consider the non-stationary case where αa is much larger than $Re_\beta^{1/2}$, or in other words to the case that the penetration depth is much smaller than the distance from the centre of the sphere to the so-called outer region used in the method of matched asymptotic expansions. On the other hand, we consider the stationary case for which the penetration depth is infinite. For the the stationary case, the value of one of the off-diagonal components which gives the so-called lift force agrees with the result of Saffman. A comparison with the results of Harper & Chang† for the stationary case is less satisfactory, a matter which is discussed in detail in §5. In the last section, we discuss our results.

2. The equations of motion and the induced force field

We consider a sphere with radius a, whose centre is located at a position $\boldsymbol{R}(t)$, moving through an incompressible fluid. The motion of the fluid surrounding the sphere is described by the Navier–Stokes equation and the continuity equation:

$$\left.\begin{aligned} \rho\frac{\partial}{\partial t}\boldsymbol{v} + \rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} &= -\nabla\cdot\boldsymbol{P}, \\ \nabla\cdot\boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{for} \quad |\,\boldsymbol{r}-\boldsymbol{R}(t)\,| > a, \tag{2.1}$$

with the stress tensor

$$P_{ij} = p\delta_{ij} - \eta\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right), \tag{2.2}$$

where ρ is the mass density, $\boldsymbol{v}$ is the velocity field, p is the hydrostatic pressure and η is the shear viscosity. On the other hand, the motion of the sphere is described by

$$m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t) = \boldsymbol{K}(t) + \boldsymbol{K}_{ext}(t) = -\int_{S(t)} \mathrm{d}\boldsymbol{n}\,\boldsymbol{P}\cdot\boldsymbol{n} + \boldsymbol{K}_{ext}(t), \tag{2.3}$$

where m is the mass of the sphere, $\boldsymbol{u}(t)$ is the velocity, $\boldsymbol{K}(t)$ is the force exerted on the sphere by the fluid and $\boldsymbol{K}_{ext}(t)$ is an external force. $\boldsymbol{K}(t)$ is given by integration of the stress tensor over the surface of the sphere at time t, $S(t)$. Here $\boldsymbol{n}$ is the unit vector pointing outward from the surface of the sphere. Those equations for the fluid and the sphere are related to each other through boundary conditions. We use stick boundary conditions at the surface of the sphere:

$$\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{u}(t) + \boldsymbol{\Omega}(t)\times(\boldsymbol{r}-\boldsymbol{R}(t)) \qquad \text{for} \quad |\,\boldsymbol{r}-\boldsymbol{R}(t)\,| = a, \tag{2.4}$$

where $\boldsymbol{\Omega}(t)$ is the angular velocity of the sphere.

In this paper, we consider the general case where, in the absence of the sphere, the fluid is in homogeneous incompressible flow given by

$$\boldsymbol{v}_0(\boldsymbol{r}) = \boldsymbol{\beta}\cdot\boldsymbol{r}, \tag{2.5}$$

where $\boldsymbol{\beta}$ is a constant traceless tensor. All the general results will be for this case.

† Notice that in their expression for F_{13} in eq.(A.28) one should replace k_3 by k_1k_3 and k_1 by k_1^2.

The explicit calculation of numerical prefactors is done, in particular, for the case of simple shear flow:

$$\beta_{ij} = \beta\delta_{iz}\delta_{jx}, \quad \text{so that} \quad \boldsymbol{v}_0(\boldsymbol{r}) = (0, 0, \beta x), \tag{2.6}$$

where β is the shear rate. In order to make the discussion clear, we choose a co-moving coordinate frame in the centre of the sphere. Then, the unperturbed velocity field is given by

$$\boldsymbol{v}_0(\boldsymbol{r}, t) = \boldsymbol{\beta}\cdot(\boldsymbol{r} + \boldsymbol{R}(t)) - \boldsymbol{u}(t), \tag{2.7}$$

where $\boldsymbol{r}$ is now the distance from the centre of the sphere. In consequence of this choice of the reference frame, the boundary condition can be rewritten as

$$\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{\Omega}(t)\times\boldsymbol{r} \qquad \text{for} \quad r = a. \tag{2.8}$$

Linearizing the Navier–Stokes equation in the perturbation caused by the presence of the sphere, $\delta\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v}_0$, one obtains

$$\left.\begin{aligned} &\rho\frac{\partial}{\partial t}\boldsymbol{v} + \rho\{\boldsymbol{\beta}\cdot(\boldsymbol{r} + \boldsymbol{R}(t)) - \mathbf{u}(t)\}\cdot\nabla\boldsymbol{v} + \rho\boldsymbol{\beta}\cdot\boldsymbol{v} = -\nabla p^* + \eta\nabla^2\boldsymbol{v} - \rho\frac{\mathrm{d}\boldsymbol{u}(t)}{\mathrm{d}t} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned}\right\} \quad \text{for } r > a, \tag{2.9}$$

with a modified pressure defined by

$$\nabla p^* \equiv \nabla p - \rho\boldsymbol{\beta}\cdot\boldsymbol{v}_0(\boldsymbol{r}, t). \tag{2.10}$$

The last term on the right-hand side of (2.9) originates from the fact that we chose the non-inertial frame as a reference frame. In the cases of simple shear flow, rotational flow and elongational flow, (2.10) may be integrated and p^* can be expressed explicitly as

$$p^* = p - \tfrac{1}{2}\rho(\boldsymbol{r} - \boldsymbol{\beta}\cdot\boldsymbol{R}(t))\cdot\boldsymbol{\beta}^2\cdot(\boldsymbol{r} - \boldsymbol{\beta}\cdot\boldsymbol{R}(t)) + \rho\boldsymbol{r}\cdot\boldsymbol{\beta}\cdot\boldsymbol{u}(t). \tag{2.11}$$

Particularly for simple shear flow, this expression reduces, using $\boldsymbol{\beta}^2 = 0$, to

$$p^* = p + \rho\boldsymbol{r}\cdot\boldsymbol{\beta}\cdot\boldsymbol{u}(t). \tag{2.12}$$

As was already pointed out by Saffman (1965) and McLaughlin (1991), if $Re_u \equiv |\Delta\boldsymbol{u}|a/\nu$ satisfies $Re_u \ll Re_\beta^{1/2}$, the Oseen term $\Delta\boldsymbol{u}(t)\cdot\nabla\boldsymbol{v}$ in (2.9) can be neglected. Here $\Delta\boldsymbol{u} \equiv \boldsymbol{u} - \boldsymbol{\beta}\cdot\boldsymbol{R}$ is the relative velocity of the sphere to the local velocity field. This can be justified as follows. There are two distinguishing length scales involved in the system: $L_u \equiv \nu/|\Delta\boldsymbol{u}|$ and $L_\beta \equiv (\nu/\beta)^{1/2}$. The condition (1.2) shows that $L_\beta \ll L_u$. At a distance of the order L_β, the inertial term $\rho(\boldsymbol{\beta}\cdot\boldsymbol{r})\cdot\nabla\boldsymbol{v} + \rho\boldsymbol{\beta}\cdot\boldsymbol{v}$ is the same order as the viscous term $\eta\nabla^2\boldsymbol{v}$, while the Oseen term is the order of $Re_u/Re_\beta^{1/2}$ compared to the inertial term, so that the Oseen term can be neglected. At a distance larger than L_β, the Oseen term becomes even smaller compared to the inertial term. On the other hand, at a distance smaller than L_β, the viscous term becomes predominant, while the Oseen term becomes small. Though the inertial term also becomes negligible in this region, one can safely keep this term in the equation. We thus find that for $Re_u \ll Re_\beta^{1/2} \ll 1$, (2.9) reduces to

$$\left.\begin{aligned} &\rho\frac{\partial}{\partial t}\boldsymbol{v} + \rho\boldsymbol{r}\cdot\boldsymbol{\beta}^\dagger\cdot\nabla\boldsymbol{v} + \rho\boldsymbol{\beta}\cdot\boldsymbol{v} = -\nabla p^* + \eta\nabla^2\boldsymbol{v} - \rho\frac{\mathrm{d}\boldsymbol{u}(t)}{\mathrm{d}t}, \\ &\nabla\cdot\boldsymbol{v} = 0, \end{aligned}\right\} \tag{2.13}$$

where $\boldsymbol{\beta}^\dagger$ stands for the transpose of $\boldsymbol{\beta}$. In the method of matched asymptotic expansions, one distinguishes an outer region, $r \gg L_\beta$, and an inner region, $r \ll L_\beta$. In the outer region, one uses a solution of the above equation for a point force in the centre of the sphere and matches it to a solution of the equation in the inner region where, as discussed above, one may neglect the terms proportional to β. In the method described below, we give a solution of (2.13) valid for all $r \geqslant a$ in terms of an integral over the force distribution on the surface of the sphere. This solution reduces to the one used in the method of matched asymptotic expansions in the outer region which is verified explicitly below. In the inner region, it also reduces to the solution in the method of matched asymptotic expansions to the first order in an expansion in $Re_\beta^{1/2}$. In our analysis below, we do not verify this fact explicitly. The method we use enables us to calculate the integral over the force density on the surface of the sphere (the total force) without the need to calculate the velocity field in the inner region explicitly. The integral representation is enough. As such our method is simpler than the method of matched asymptotic expansions. One of the reasons for verifying in such detail that our method leads to the same result as the method of matched asymptotic expansions for simple shear is to establish that the two methods are in fact equivalent.

Using the induced force method (Mazur & Bedeaux 1974; Bedeaux & Rubi 1987), the linearized Navier–Stokes equation, (2.13), is extended inside the sphere and is reformulated for all $\boldsymbol{r}$ as

$$\left.\begin{aligned} &\rho\frac{\partial}{\partial t}\boldsymbol{v} + \rho\boldsymbol{r}\cdot\boldsymbol{\beta}^\dagger\cdot\nabla\boldsymbol{v} + \rho\boldsymbol{\beta}\cdot\boldsymbol{v} = -\nabla p^* + \eta\nabla^2\boldsymbol{v} - \rho\frac{\mathrm{d}\boldsymbol{u}(t)}{\mathrm{d}t} + \boldsymbol{F}_{ind}, \\ &\nabla\cdot\boldsymbol{v} = 0. \end{aligned}\right\} \tag{2.14}$$

Here $\boldsymbol{F}_{ind}$ is the induced force field introduced in order to take the forces exerted by the sphere on the fluid into account. As (2.13) holds outside the sphere, one has

$$\boldsymbol{F}_{ind}(\boldsymbol{r}, t) = 0 \qquad \text{for} \quad r > a. \tag{2.15}$$

Inside the sphere, the induced force is chosen such that

$$\left.\begin{aligned} &\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{\Omega}(t)\times\boldsymbol{r} \\ &p = 0 \end{aligned}\right\} \qquad \text{for} \quad r \leqslant a. \tag{2.16}$$

There is freedom of choice of the field variables inside the sphere. A different choice is compensated by a corresponding modification of the induced force. Physical quantities like, for instance, the force on the sphere are independent of the particular choice. Here we made a choice such that the velocity field is continuous at the surface of the sphere. In order to make (2.14) consistent inside the sphere, the induced force field has to satisfy

$$\begin{aligned} \boldsymbol{F}_{ind}(\boldsymbol{r}, t) &= \rho\left(\frac{\partial}{\partial t} + \boldsymbol{r}\cdot\boldsymbol{\beta}^\dagger\cdot\nabla + \boldsymbol{\beta}\right)\cdot(\boldsymbol{\Omega}(t)\times\boldsymbol{r}) - \rho\boldsymbol{\beta}\cdot\boldsymbol{v}_0(\boldsymbol{r}, t) + \rho\frac{\mathrm{d}\boldsymbol{u}(t)}{\mathrm{d}t} \\ &\equiv \boldsymbol{F}_{inner}(\boldsymbol{r}, t) \qquad\qquad \text{for} \quad r < a. \end{aligned} \tag{2.17}$$

Including a possible singular contribution on the surface of the sphere, the total induced force has the general form

$$\boldsymbol{F}_{ind}(\boldsymbol{r}, t) = \boldsymbol{F}_s(\boldsymbol{n}, t)\delta(a - r) + \boldsymbol{F}_{inner}(\boldsymbol{r}, t) \qquad \text{for} \quad r \leqslant a, \tag{2.18}$$

where $\boldsymbol{F}_s(\boldsymbol{n}, t)$ is the induced force density per unit area of the surface of the sphere.

Substituting (2.14) into (2.3) and using (2.16), (2.17) and (2.18), the force exerted on the sphere by the fluid, $\boldsymbol{K}(t)$, can be related to the induced force density by

$$\begin{aligned}\boldsymbol{K}(t) &= -\int_{S(t)} \mathrm{d}\boldsymbol{n}\,\boldsymbol{P}\cdot\boldsymbol{n} = -\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\,\nabla\cdot\boldsymbol{P} \\ &= \int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\left\{\rho\left(\frac{\partial}{\partial t}\boldsymbol{v} + \boldsymbol{r}\cdot\boldsymbol{\beta}^{\dagger}\cdot\nabla\boldsymbol{v} + \boldsymbol{\beta}\cdot\boldsymbol{v}\right) - \rho\boldsymbol{\beta}\cdot\boldsymbol{v}_0(\boldsymbol{r},t) + \rho\frac{\mathrm{d}\boldsymbol{u}(t)}{\mathrm{d}t} - \boldsymbol{F}_{ind}\right\} \\ &= \int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\,\{\boldsymbol{F}_{inner} - \boldsymbol{F}_{ind}\} \\ &= -a^2\int \mathrm{d}\boldsymbol{n}\,\boldsymbol{F}_s(\boldsymbol{n},t). \end{aligned} \tag{2.19}$$

As we shall show in the subsequent sections, it is sufficient to construct a formal solution of (2.14) using a Green function in order to obtain $\boldsymbol{K}(t)$. It is not necessary to construct an explicit solution for the velocity field around the sphere.

3. General solution

In order to solve (2.14), we introduce Fourier transformation in space, e.g. for the velocity field

$$\boldsymbol{v}(\boldsymbol{k},t) = \int \mathrm{d}\boldsymbol{r}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{v}(\boldsymbol{r},t). \tag{3.1}$$

The equations of motion, (2.14), then become

$$\left.\begin{aligned} &\rho\left(\frac{\partial}{\partial t} + \nu k^2 + \boldsymbol{\beta} - \boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial \boldsymbol{k}}\right)\cdot\boldsymbol{v}(\boldsymbol{k},t) = -\mathrm{i}\boldsymbol{k}p^{*} - \rho\frac{\mathrm{d}\boldsymbol{u}(t)}{\mathrm{d}t}(2\pi)^3\delta(\boldsymbol{k}) + \boldsymbol{F}_{ind}(\boldsymbol{k},t), \\ &\boldsymbol{k}\cdot\boldsymbol{v} = 0, \end{aligned}\right\} \tag{3.2}$$

where $\nu = \eta/\rho$ is the kinematic viscosity. The formal solution of these equations is given by

$$\boldsymbol{v}(\boldsymbol{k},t) = \boldsymbol{v}_0(\boldsymbol{k},t) + \int_{-\infty}^{t} \mathrm{d}t'\,\frac{1}{\rho}\exp\left[-(t-t')\left\{\mathbf{g}(\boldsymbol{k}) - \boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial \boldsymbol{k}}\right\}\right]\cdot\boldsymbol{P}_k\cdot\boldsymbol{F}_{ind}(\boldsymbol{k},t') \tag{3.3}$$

with

$$\mathbf{g}(\boldsymbol{k}) = \nu k^2 + (\mathbf{1} - 2\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot\boldsymbol{\beta}, \tag{3.4}$$

where $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/|\boldsymbol{k}|$ and

$$\boldsymbol{P}_k = \mathbf{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}} \tag{3.5}$$

is the transverse projection operator. For the derivation of (3.3), refer to Appendix A. Note that, owing to the presence of the term containing a differential operator in $\boldsymbol{k}$, the operator acting on the homogeneous term in (3.3) is not diagonal in the $\boldsymbol{k}$-representation. Bedeaux & Rubi (1987) and Pérez-Madrid *et al.* (1990) neglected this term.

Following Onuki & Kawasaki (1980) †, we make use of the relations

$$\exp\left[t\boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right]\boldsymbol{f}(\boldsymbol{k})=\boldsymbol{f}(\boldsymbol{k}(t)), \tag{3.6}$$

$$\exp\left[-t\mathbf{g}(\boldsymbol{k})+t\boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right]\cdot\exp\left[-t\boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right]=\mathscr{T}_{\leftarrow}\exp\left[-\int_0^t\mathrm{d}\tau\,\mathbf{g}(\boldsymbol{k}(\tau))\right], \tag{3.7}$$

where $\mathscr{T}_{\leftarrow}$ is the time-ordering operator, defined by

$$\begin{aligned}\mathscr{T}_{\leftarrow}\exp&\left[-\int_0^t\mathrm{d}\tau\,\mathbf{g}(\boldsymbol{k}(\tau))\right]\\&\equiv\mathbf{1}-\int_0^t\mathrm{d}\tau_1\,\mathbf{g}(\boldsymbol{k}(\tau_1))+\int_0^t\mathrm{d}\tau_1\int_{\tau_1}^t\mathrm{d}\tau_2\,\mathbf{g}(\boldsymbol{k}(\tau_1))\cdot\mathbf{g}(\boldsymbol{k}(\tau_2))-\cdots\ .\end{aligned} \tag{3.8}$$

Here we have introduced the time-dependent wavevector defined by

$$\boldsymbol{k}(t)\equiv\exp\left[\boldsymbol{\beta}^{\dagger}\,t\right]\cdot\boldsymbol{k}. \tag{3.9}$$

Then, (3.3) becomes

$$\boldsymbol{v}(\boldsymbol{k},t)=\boldsymbol{v}_0(\boldsymbol{k},t)+\int_{-\infty}^t\mathrm{d}t'\,\mathbf{G}(\boldsymbol{k},t-t')\cdot\boldsymbol{F}_{ind}(\boldsymbol{k}(t-t'),t'), \tag{3.10}$$

where the Green function is defined by

$$\mathbf{G}(\boldsymbol{k},t)=\frac{1}{\rho}\mathscr{T}_{\leftarrow}\exp\left[-\int_0^t\mathrm{d}\tau\,\mathbf{g}(\boldsymbol{k}(\tau))\right]\cdot\mathbf{P}_{\boldsymbol{k}(t)}. \tag{3.11}$$

By Fourier transformation of (3.10) back to the $\boldsymbol{r}$-representation, one obtains

$$\boldsymbol{v}(\boldsymbol{r},t)=\boldsymbol{v}_0(\boldsymbol{r},t)+\int_{-\infty}^t\mathrm{d}t'\int\mathrm{d}\boldsymbol{r}'\,\mathbf{G}(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot\boldsymbol{F}_{ind}(\boldsymbol{r}(t-t'),t'), \tag{3.12}$$

where

$$\mathbf{G}(\boldsymbol{r},t)=\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathbf{G}(\boldsymbol{k},t) \tag{3.13}$$

and

$$\boldsymbol{r}(t)\equiv\exp\left[-\boldsymbol{\beta}t\right]\cdot\boldsymbol{r}. \tag{3.14}$$

Equation (3.12) is the general solution for (2.14). This expression is valid for an arbitrary constant matrix, $\boldsymbol{\beta}$, in so far as the governing equation is given by (2.14). Equation (3.14) stands for the transformation to the co-moving frame which is moving along with the local unperturbed velocity field. In the case of simple shear flow, the above equation reduces to the Galilean transformation

$$\boldsymbol{r}(t)=\boldsymbol{r}-\boldsymbol{\beta}\cdot\boldsymbol{r}\,t. \tag{3.15}$$

The velocity field is related to the induced force at all previous times and at all the positions through the Green function. However, the position of the induced force has to be replaced by a co-moving one because the force at time t' and at a position $\boldsymbol{r}$ which is supposed to affect the velocity field flows along with the local velocity field, so to say, and will be located at $\boldsymbol{r}(t-t')$ at time t.

† They used the relation (3.6) and (3.7) for a scalar variable. In our case, however, as the variables are vectors, the time-ordering operator has to be introduced.

Furthermore, note that the solution given above has following translational invariance:

$$\begin{aligned}\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{r}_0,t) &= \boldsymbol{v}_0(\boldsymbol{r}+\boldsymbol{r}_0,t)\\ &+\int_{-\infty}^{t}\mathrm{d}t'\int\mathrm{d}\boldsymbol{r}'\,\mathbf{G}(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot\boldsymbol{F}_{ind}(\boldsymbol{r}'(t-t')+\boldsymbol{r}_0(t-t'),t'). \end{aligned}\qquad (3.16)$$

Thus, a shift of $\boldsymbol{F}_{ind}$ over $\boldsymbol{r}_0$ leads to a corresponding shift of the velocity field, $\boldsymbol{v}-\boldsymbol{v}_0$. Though this should be satisfied for obvious reasons, it is good to see that the general solution has this property in such a clear manner.

4. The mobility tensor

By using the result of the previous section and (2.18), one obtains a one-to-one relation between the velocity field at the surface of the sphere and the induced force in the frequency representation in the following form:

$$\boldsymbol{v}(a\boldsymbol{n},\omega)=\boldsymbol{v}_0(a\boldsymbol{n},\omega)+\int\mathrm{d}\boldsymbol{n}'\,\boldsymbol{M}(\boldsymbol{n},\omega\mid\boldsymbol{n}')\cdot\boldsymbol{F}_s(\boldsymbol{n}',\omega)+\boldsymbol{f}(a\boldsymbol{n},\omega). \qquad (4.1)$$

Here we have used that a position at the surface of the sphere is given by $a\boldsymbol{n}$ in terms of the unit vector pointing outward on the surface. Furthermore

$$\boldsymbol{M}(\boldsymbol{n},\omega\mid\boldsymbol{n}')\equiv a^2\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int_0^\infty\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot a\boldsymbol{n}}\mathbf{G}(\boldsymbol{k},t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}(t)\cdot a\boldsymbol{n}'} \qquad (4.2)$$

is the coefficient connecting the velocity field at the surface to the induced force density and

$$\boldsymbol{f}(a\boldsymbol{n},\omega)\equiv\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int_0^\infty\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot a\boldsymbol{n}}\mathbf{G}(\boldsymbol{k},t)\cdot\boldsymbol{F}_{inner}(\boldsymbol{k}(t),\omega) \qquad (4.3)$$

is the contribution from the induced force inside the sphere. One may expand a function of $\boldsymbol{n}$ in terms of a complete set of irreducible tensors (see, for example, Mazur & van Saarloos 1982; Mazur & Weisenborn 1984) in the following form:

$$\alpha(\boldsymbol{n})=\sum_{l=0}^{\infty}\frac{(2l+1)!!}{l!}\overset{\sqcap}{\boldsymbol{n}}{}^{l}\odot\boldsymbol{\alpha}(l) \qquad (4.4)$$

with a multipole coefficient

$$\boldsymbol{\alpha}(l)\equiv\frac{1}{4\pi}\int\mathrm{d}\boldsymbol{n}\,\overset{\sqcap}{\boldsymbol{n}}{}^{l}\alpha(\boldsymbol{n}). \qquad (4.5)$$

Here $\overset{\sqcap}{\boldsymbol{n}}{}^{l}$ stands for the symmetric and traceless tensor of rank l constructed with $\boldsymbol{n}$. The symbol $\odot$ denotes the full contraction of the tensor $\overset{\sqcap}{\boldsymbol{n}}{}^{l}$ and the coefficient of rank l, $\boldsymbol{\alpha}(l)$. By using these formulae and (4.1), one finds the following set of linear equations for the multipole coefficients of an arbitrary rank:

$$\boldsymbol{v}(\omega,l)=\boldsymbol{v}_0(\omega,l)+\sum_{l'=0}^{\infty}\frac{(2l'+1)!!}{l'!}\boldsymbol{M}(\omega,l\mid l')\odot\boldsymbol{F}_s(\omega,l')+\boldsymbol{f}(\omega,l). \qquad (4.6)$$

Here we define the $(l+l'+2)$-fold matrix, $\boldsymbol{M}(\omega,l\mid l')$, by

$$\boldsymbol{M}(\omega,l\mid l')\equiv\frac{1}{4\pi}\int\mathrm{d}\boldsymbol{n}\int\mathrm{d}\boldsymbol{n}'\,\overset{\sqcap}{\boldsymbol{n}}{}^{l}\boldsymbol{M}(\boldsymbol{n},\omega\mid\boldsymbol{n}')\,\overset{\sqcap}{\boldsymbol{n}'}{}^{l'}. \qquad (4.7)$$

According to (2.19), the force exerted on the sphere, $\boldsymbol{K}(\omega)$, is related to the first multipole coefficient of the induced force density by

$$\boldsymbol{K}(\omega) = -4\pi a^2 \boldsymbol{F}_s(\omega, l=0). \tag{4.8}$$

Equation (4.6) can be solved in terms of the first multipole coefficient of the induced force density formally as follows (Mazur & van Saarloos 1982):

$$\begin{aligned}
&\boldsymbol{v}(\omega,0) - \boldsymbol{v}_0(\omega,0) - \boldsymbol{f}(\omega,0) \\
&\quad = \boldsymbol{M}(\omega,0 \mid 0) \odot \boldsymbol{F}_s(\omega,0) + \sum_{j=0}^{\infty}(-1)^j \sum_{m_1=1}^{\infty} \cdots \sum_{m_j=1, m_j \neq m_{j-1}}^{\infty} \\
&\quad \boldsymbol{M}(\omega,0 \mid m_1) \odot \boldsymbol{M}^{-1}(\omega,m_1 \mid m_1) \odot \boldsymbol{M}(\omega,m_1 \mid m_2) \odot \cdots \boldsymbol{M}^{-1}(\omega,m_j \mid m_j) \odot \\
&\quad \left[\boldsymbol{M}(\omega,m_j \mid 0) \odot \boldsymbol{F}_s(\omega,0) + \boldsymbol{v}(\omega,m_j) - \boldsymbol{v}_0(\omega,m_j) - \boldsymbol{f}(\omega,m_j)\right].
\end{aligned} \tag{4.9}$$

It can be easily shown that $\boldsymbol{M}(\omega, l \mid l') = 0$ for $l + l' =$ odd by using the symmetric property of the irreducible tensors and the Green function. Furthermore, one finds, using (2.7), (2.16), and (4.3),

$$v_{i,j_1 j_2 \cdots j_l}(\omega, l) = \tfrac{1}{3} a \epsilon_{i j_1 k} \Omega_k \delta_{l1}, \tag{4.10}$$

$$\boldsymbol{v}_0(\omega, l) = [\boldsymbol{\beta} \cdot \boldsymbol{R}(\omega) - \boldsymbol{u}(\omega)]\, \delta_{l0} + \tfrac{1}{3} a \boldsymbol{\beta} \delta_{l1}, \tag{4.11}$$

$$\boldsymbol{f}(\omega, l) = \boldsymbol{f}(\omega, 0)\delta_{l0} + \boldsymbol{f}(\omega, 1)\delta_{l1}, \tag{4.12}$$

where ϵ_{ijk} is the Levi–Civita tensor.

For $\boldsymbol{M}(\omega, l \mid l')$ with $|l - l'| =$ even, the following relation may be shown to be valid:

$$\boldsymbol{M}(\omega, l \mid l') = O\left(\left\{Re_\beta^{1/2}\ d(\omega/\beta)\ \right\}^{|l-l'|}\right), \tag{4.13}$$

where $d(x)$ is a function which has the asymptotic value $(-\mathrm{i}x)^{1/2}$ for large x and which approaches a constant for small x. Thus if ω is large enough compared to β, one can replace $Re_\beta^{1/2}\ d(\omega/\beta)$ in (4.13) by $\alpha a = (-\mathrm{i}\omega a^2/\nu)^{1/2}$. The derivation of (4.13) is along the same lines as a similer derivation given by Mazur & Weisenborn (1984) for the calculation of the Oseen drag on a sphere and, as it is rather long and not very illuminating, it will not be given explicitly. Notice the fact that the analysis in this paper is restricted to cases such that the penetration depth is large compared to the radius of the sphere, i.e. $\alpha a \ll 1$. The important conclusion is that for $\omega > \beta$ and for $\omega < \beta$, i.e. both for a penetration depth either smaller than or larger than $L_\beta \equiv (\nu/\beta)^{1/2}$, the matrix elements $\boldsymbol{M}(\omega, l \mid l')$ with $\mid l - l' \mid \geqslant 2$ in (4.9) may be neglected. We note that in the following section we will calculate the friction coefficient for both $\omega \gg \beta$ and $\omega \to 0$, i.e. both for a penetration depth much smaller than L_β and when the penetration depth is infinite.

Also, the contribution from the zeroth coefficient of the induced force inside the sphere is given, up to the lowest order, as

$$\boldsymbol{f}(\omega, 0) = \frac{2a^2}{9\nu}\left[(-\mathrm{i}\omega + \boldsymbol{\beta}) \cdot \boldsymbol{u}(\omega) - \boldsymbol{\beta}^2 \cdot \boldsymbol{R}(\omega)\right]. \tag{4.14}$$

The first term in the square bracket on the right-hand side is an inertial effect of the excluded fluid and the second one is the buoyancy force caused by the gradient of the hydrostatic pressure. As these terms are the second order in αa or $Re_\beta^{1/2}$, one can neglect them as well. Other non-zero quantities, $\boldsymbol{v}(\omega, 1)$, $\boldsymbol{v}_0(\omega, 1)$ and $\boldsymbol{f}(\omega, 1)$, do not contribute because the coefficient $\boldsymbol{M}(\omega,0 \mid 1)$ is zero. Then up to the first order in $Re_\beta^{1/2}$, one obtains the simplified relation

$$\boldsymbol{F}_s(\omega,0) = \boldsymbol{M}(\omega,0 \mid 0)^{-1} \odot \{\boldsymbol{v}(\omega,0) - \boldsymbol{v}_0(\omega,0)\}. \tag{4.15}$$

By substituting this equation to (4.8), one obtains

$$\boldsymbol{u}(\omega) - \boldsymbol{\beta}\cdot\boldsymbol{R}(\omega) = -\boldsymbol{\mu}(\omega)\cdot\boldsymbol{K}(\omega), \tag{4.16}$$

where we used (4.10) and (4.11). In this expression, $\boldsymbol{\mu}(\omega)$ is the mobility tensor which is given by

$$\begin{aligned}\boldsymbol{\mu}(\omega) &= \frac{1}{4\pi a^2}\boldsymbol{M}(\omega,0 \mid 0)\\ &= \frac{1}{(4\pi a)^2}\int \mathrm{d}\boldsymbol{n}\int \mathrm{d}\boldsymbol{n}'\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot a\boldsymbol{n}}\mathbf{G}(\boldsymbol{k},t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}(t)\cdot a\boldsymbol{n}'}\\ &= \int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\frac{\sin ka}{ka}\mathbf{G}(\boldsymbol{k},t)\frac{\sin k(t)a}{k(t)a}.\end{aligned} \tag{4.17}$$

This together with the solution of the Navier–Stokes equation, (3.12), is the central result of this paper. It shows that the mobility tensor can be expressed in terms of the Green function. In this expression, the time-dependence is also taken into account. One should notice that we did not specify $\boldsymbol{\beta}$ during the derivation of (4.17), so that it can be applied for arbitrary homogeneous flow in so far as the flow is incompressible and can be written as a linear function of the position. In the next section, we will calculate the mobility tensor for the simple shear case using the above expression. We will reproduce Saffman's result for the lift force. It will turn out that, if one replaces the induced force, $\boldsymbol{F}_{ind}$, by a point force which is given by

$$\boldsymbol{F}_{ind}(\boldsymbol{r},t) = -\boldsymbol{K}(\omega)\delta(\boldsymbol{r}), \tag{4.18}$$

then the explicit expression given in (4.17) for the x,z-component of the mobility for simple shear is simplified because $\sin ka/ka$ and $\sin k(t)a/k(t)a$ are then replaced by unity and (4.17) reduces to the one Saffman derived, (5.25) below, in a much more direct way. Note in this context that, in an analysis given by McLaughlin, the lift force was also calculated using a velocity originating from the point force defined by (4.18). Though this replacement is found to give the exact result by explicit calculations and for obvious reasons it gives the correct velocity field far from the sphere, it is nevertheless somewhat surprising that (4.17), which is found by averaging the velocity field close to the sphere, may be reduced in this manner. This fact is, however, very much in line with Saffman's conclusions regarding the limited role of the inner expansion of the velocity field.

5. Simple shear case

We have derived the frequency-dependent mobility tensor which is applicable to arbitrary stationary homogeneous flow and is valid up to the lowest order in $Re_\beta^{1/2}$ and αa in the previous section. In this section, we restrict ourselves to simple shear flow. As it is not possible to give the analytical form of the mobility for an arbitrary frequency, we focus on two particular cases in the following: the case where $\omega \gg \beta$ and the stationary case, i.e. $\omega = 0$.

5.1. *The case where $\omega \gg \beta$*

In this regime, L_β is much larger than the penetration depth which is much larger than the radius of the sphere. We introduce two small dimensionless parameters

$$\frac{\beta}{\omega} \equiv \epsilon_1, \qquad \frac{\omega a^2}{\nu} = \mathrm{i}(\alpha a)^2 \equiv \epsilon_2. \tag{5.1}$$

In order to make the discussion clear, we introduce dimensionless variables defined by

$$\omega t \equiv \tilde{t}, \quad \omega\tau \equiv \tilde{\tau}, \quad ka \equiv \tilde{\boldsymbol{k}}. \tag{5.2}$$

Then the mobility tensor, (4.17), is written as

$$\begin{aligned} \boldsymbol{\mu}(\omega) = \frac{1}{a\eta\epsilon_2}\int\frac{\mathrm{d}\tilde{\boldsymbol{k}}}{(2\pi)^3}\int_0^\infty \mathrm{d}\tilde{t}\,\mathrm{e}^{\mathrm{i}\tilde{t}}\frac{\sin\tilde{k}}{\tilde{k}}\frac{\sin c(\hat{\boldsymbol{k}},\tilde{t})\tilde{k}}{c(\hat{\boldsymbol{k}},\tilde{t})\tilde{k}}\exp\left[-D(\hat{\boldsymbol{k}},\tilde{t})\tilde{k}^2/\epsilon_2\right] \\ \times\mathscr{T}_{\leftarrow}\exp\left[-\epsilon_1\int_0^{\tilde{t}}\mathrm{d}\tilde{\tau}\,\boldsymbol{A}(\hat{\boldsymbol{k}},\tilde{\tau})\cdot\boldsymbol{B}\right]\cdot\left(\boldsymbol{1}-\frac{1}{c^2(\hat{\boldsymbol{k}},\tilde{t})}\mathrm{e}^{\epsilon_1\boldsymbol{B}^\dagger\tilde{t}}\cdot\hat{\boldsymbol{k}}\;\mathrm{e}^{\epsilon_1\boldsymbol{B}^\dagger\tilde{t}}\cdot\hat{\boldsymbol{k}}\right), \end{aligned} \tag{5.3}$$

where

$$c(\hat{\boldsymbol{k}},t) = \left(1+2\epsilon_1\hat{k}_x\hat{k}_z t+\epsilon_1^2\hat{k}_z^2t^2\right), \tag{5.4}$$

$$D(\hat{\boldsymbol{k}},t) = t\left(1+\epsilon_1\hat{k}_x\hat{k}_z t+\epsilon_1^2\hat{k}_z^2t^2/3\right), \tag{5.5}$$

$$\boldsymbol{A}(\hat{\boldsymbol{k}},t) = \boldsymbol{1}-\frac{2}{c^2(\hat{\boldsymbol{k}},t)}\mathrm{e}^{\epsilon_1\boldsymbol{B}^\dagger t}\cdot\hat{\boldsymbol{k}}\;\mathrm{e}^{\epsilon_1\boldsymbol{B}^\dagger t}\cdot\hat{\boldsymbol{k}}, \tag{5.6}$$

$$B_{ij} = \delta_{iz}\delta_{jx}. \tag{5.7}$$

Henceforth tildes will be omitted for simplification. After integrating over k, one arrives at

$$\begin{aligned} \boldsymbol{\mu}(\omega) = \frac{1}{8\pi^{3/2}a\eta\epsilon_2^{1/2}} \\ \times\int\frac{\mathrm{d}\hat{\boldsymbol{k}}}{4\pi}\int_0^\infty\mathrm{d}t\,\frac{1}{c(\hat{\boldsymbol{k}},t)D^{1/2}(\hat{\boldsymbol{k}},t)}\exp\left[-\frac{\epsilon_2\left\{c(\hat{\boldsymbol{k}},t)-1\right\}^2}{4D(\hat{\boldsymbol{k}},t)}\right]\left(1-\exp\left[-\frac{\epsilon_2 c(\hat{\boldsymbol{k}},t)}{D(\hat{\boldsymbol{k}},t)}\right]\right)\mathrm{e}^{\mathrm{i}t} \\ \times\mathscr{T}_{\leftarrow}\exp\left[-\int_0^t\mathrm{d}\tau\,\boldsymbol{A}(\hat{\boldsymbol{k}},\tau)\cdot\boldsymbol{B}\right]\cdot\left(\boldsymbol{1}-\frac{1}{c^2(\hat{\boldsymbol{k}},t)}\mathrm{e}^{\epsilon_1\boldsymbol{B}^\dagger t}\cdot\hat{\boldsymbol{k}}\;\mathrm{e}^{\epsilon_1\boldsymbol{B}^\dagger t}\cdot\hat{\boldsymbol{k}}\right). \end{aligned} \tag{5.8}$$

Expanding the integrand in ϵ_1 up to the first order, one obtains

$$\begin{aligned} \frac{1}{c(\hat{\boldsymbol{k}},t)D^{1/2}(\hat{\boldsymbol{k}},t)}\exp\left[-\frac{\epsilon_2\left\{c(\hat{\boldsymbol{k}},t)-1\right\}^2}{4D(\hat{\boldsymbol{k}},t)}\right]\left(1-\exp\left[-\frac{\epsilon_2 c(\hat{\boldsymbol{k}},t)}{D(\hat{\boldsymbol{k}},t)}\right]\right)\mathrm{e}^{\mathrm{i}t} \\ \times\mathscr{T}_{\leftarrow}\exp\left[-\int_0^t\mathrm{d}\tau\,\boldsymbol{A}(\hat{\boldsymbol{k}},\tau)\cdot\boldsymbol{B}\right]\cdot\left(\boldsymbol{1}-\frac{1}{c^2(\hat{\boldsymbol{k}},t)}\mathrm{e}^{\epsilon_1\boldsymbol{B}^\dagger t}\cdot\hat{\boldsymbol{k}}\;\mathrm{e}^{\epsilon_1\boldsymbol{B}^\dagger t}\cdot\hat{\boldsymbol{k}}\right) \\ \approx\frac{1}{t^{1/2}}\left(1-\mathrm{e}^{-\epsilon_2/t}\right)\mathrm{e}^{\mathrm{i}t}\left[\boldsymbol{P}_k+\left\{-\tfrac{3}{4}\hat{\boldsymbol{k}}\cdot(\boldsymbol{B}+\boldsymbol{B}^\dagger)\cdot\hat{\boldsymbol{k}}\boldsymbol{P}_k-\boldsymbol{B}\cdot\boldsymbol{P}_k+\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\cdot\boldsymbol{B}-\boldsymbol{B}^\dagger\cdot\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\right\}\epsilon_1 t\right]. \end{aligned} \tag{5.9}$$

After integrating over $\hat{\boldsymbol{k}}$ and t, one obtains for the mobility

$$\boldsymbol{\mu}(\omega) \approx \frac{1}{6\pi a\eta}\left[\mathbf{1} - \alpha a\left(\mathbf{1} + \frac{7}{40}\frac{\boldsymbol{\beta}+\boldsymbol{\beta}^{\dagger}}{-\mathrm{i}\omega}\right)\right]. \tag{5.10}$$

This result may in fact be shown to be valid for an arbitrary $\boldsymbol{\beta}$ and not only for simple shear. Equation (5.10) shows that off-diagonal components appear which are symmetric and proportional to β/ω in addition to the well-known frequency-dependent diagonal contribution given already by Stokes (1851).

5.2. *The stationary case, $\omega = 0$*

In the stationary limit, the penetration depth becomes infinite and as such much larger than L_β which is much larger than the radius of the sphere. We define dimensionless variables by

$$\beta t \equiv \tilde{t}, \quad \beta\tau \equiv \tilde{\tau}, \quad \boldsymbol{k}a \equiv \tilde{\boldsymbol{k}}. \tag{5.11}$$

Then, (4.17) is written, for $\omega = 0$, as

$$\begin{aligned}\boldsymbol{\mu}(0) = {} & \frac{1}{a\eta Re_\beta}\int\frac{\mathrm{d}\tilde{\boldsymbol{k}}}{(2\pi)^3}\int_0^\infty \mathrm{d}\tilde{t}\,\frac{\sin\tilde{k}}{\tilde{k}}\frac{\sin c(\hat{\boldsymbol{k}},\tilde{t})\tilde{k}}{c(\hat{\boldsymbol{k}},\tilde{t})\tilde{k}}\exp\left[-D(\hat{\boldsymbol{k}},\tilde{t})\tilde{k}^2/Re_\beta\right] \\ & \times\mathscr{T}_{\leftarrow}\exp\left[-\int_0^{\tilde{t}}\mathrm{d}\tilde{\tau}\,\boldsymbol{A}(\hat{\boldsymbol{k}},\tilde{\tau})\cdot\boldsymbol{B}\right]\cdot\left(\mathbf{1} - \frac{1}{c^2(\hat{\boldsymbol{k}},\tilde{t})}\mathrm{e}^{\boldsymbol{B}^{\dagger}\tilde{t}}\cdot\hat{\boldsymbol{k}}\;\mathrm{e}^{\boldsymbol{B}^{\dagger}\tilde{t}}\cdot\hat{\boldsymbol{k}}\right),\end{aligned} \tag{5.12}$$

where

$$c(\hat{\boldsymbol{k}},t) = \left(1 + 2\hat{k}_x\hat{k}_z t + \hat{k}_z^2 t^2\right), \tag{5.13}$$

$$D(\hat{\boldsymbol{k}},t) = t\left(1 + \hat{k}_x\hat{k}_z t + \hat{k}_z^2 t^2/3\right), \tag{5.14}$$

$$\boldsymbol{A}(\hat{\boldsymbol{k}},t) = \mathbf{1} - \frac{2}{c^2(\hat{\boldsymbol{k}},t)}\mathrm{e}^{\boldsymbol{B}^{\dagger}t}\cdot\hat{\boldsymbol{k}}\;\mathrm{e}^{\boldsymbol{B}^{\dagger}t}\cdot\hat{\boldsymbol{k}}, \tag{5.15}$$

$$B_{ij} = \delta_{iz}\delta_{jx}. \tag{5.16}$$

Henceforth, tildes will be omitted for simplification. After integrating over k, one arrives at

$$\begin{aligned}\boldsymbol{\mu}(0) = {} & \frac{1}{8\pi^{3/2}a\eta Re_\beta^{1/2}} \\ & \times\int\frac{\mathrm{d}\hat{\boldsymbol{k}}}{4\pi}\int_0^\infty\mathrm{d}t\,\frac{1}{c(\hat{\boldsymbol{k}},t)D^{1/2}(\hat{\boldsymbol{k}},t)}\exp\left[-\frac{Re_\beta\left\{c(\hat{\boldsymbol{k}},t)-1\right\}^2}{4D(\hat{\boldsymbol{k}},t)}\right]\left(1-\exp\left[-\frac{Re_\beta c(\hat{\boldsymbol{k}},t)}{D(\hat{\boldsymbol{k}},t)}\right]\right) \\ & \times\mathscr{T}_{\leftarrow}\exp\left[-\int_0^t\mathrm{d}\tau\,\boldsymbol{A}(\hat{\boldsymbol{k}},\tau)\cdot\boldsymbol{B}\right]\cdot\left(\mathbf{1}-\frac{1}{c^2(\hat{\boldsymbol{k}},t)}\mathrm{e}^{\boldsymbol{B}^{\dagger}t}\cdot\hat{\boldsymbol{k}}\;\mathrm{e}^{\boldsymbol{B}^{\dagger}t}\cdot\hat{\boldsymbol{k}}\right).\end{aligned} \tag{5.17}$$

Using the property (see, for example, Rabin, Wang & Creamer 1989)

$$\begin{aligned}&\boldsymbol{A}(\hat{\boldsymbol{k}},\tau_1)\cdot\boldsymbol{B}\cdot\boldsymbol{A}(\hat{\boldsymbol{k}},\tau_2)\cdot\boldsymbol{B}\cdots\boldsymbol{A}(\hat{\boldsymbol{k}},\tau_n)\cdot\boldsymbol{B} \\ &\qquad = A_{xz}(\hat{\boldsymbol{k}},\tau_2)A_{xz}(\hat{\boldsymbol{k}},\tau_3)\cdots A_{xz}(\hat{\boldsymbol{k}},\tau_n)\boldsymbol{A}(\hat{\boldsymbol{k}},\tau_1)\cdot\boldsymbol{B},\end{aligned} \tag{5.18}$$

the term involved with the time-ordering can be calculated explicitly and given by

$$\mathscr{T}_{\leftarrow}\exp\left[-\int_0^t \mathrm{d}\tau\, \boldsymbol{A}(\hat{\boldsymbol{k}},\tau)\cdot\boldsymbol{B}\right] = \boldsymbol{1} - \int_0^t \mathrm{d}\tau_1\, \boldsymbol{A}(\hat{\boldsymbol{k}},\tau_1)\cdot\boldsymbol{B}\exp\left[-\int_{\tau_1}^t d\tau_2\, A_{xz}(\hat{\boldsymbol{k}},\tau_2)\right]$$
$$= \begin{pmatrix} f_{xx}(\hat{\boldsymbol{k}},t) & 0 & 0 \\ f_{yx}(\hat{\boldsymbol{k}},t) & 1 & 0 \\ f_{zx}(\hat{\boldsymbol{k}},t) & 0 & 1 \end{pmatrix}, \tag{5.19}$$

where

$$f_{xx}(\hat{\boldsymbol{k}},t) = c^2(\hat{\boldsymbol{k}},t), \tag{5.20}$$

$$f_{yx}(\hat{\boldsymbol{k}},t) = \frac{\hat{k}_y c^2(\hat{\boldsymbol{k}},t)}{1-\hat{k}_x^2}\left[\frac{\hat{k}_x+\hat{k}_z t}{c^2(\hat{\boldsymbol{k}},t)} - \hat{k}_x + \hat{k}_z I(\hat{\boldsymbol{k}},t)\right], \tag{5.21}$$

$$f_{zx}(\hat{\boldsymbol{k}},t) = \frac{c^2(\hat{\boldsymbol{k}},t)}{1-\hat{k}_x^2}\left[\frac{\hat{k}_x\hat{k}_z+\hat{k}_z^2 t}{c^2(\hat{\boldsymbol{k}},t)} - \hat{k}_x\hat{k}_z - \hat{k}_y^2 I(\hat{\boldsymbol{k}},t)\right] \tag{5.22}$$

with

$$I(\hat{\boldsymbol{k}},t) = \frac{1}{\hat{k}_z(1-\hat{k}_x^2)^{1/2}}\left\{\arctan\left(\frac{\hat{k}_x+\hat{k}_z t}{(1-\hat{k}_x^2)^{1/2}}\right) - \arctan\left(\frac{\hat{k}_x}{(1-\hat{k}_x^2)^{1/2}}\right)\right\}. \tag{5.23}$$

Substituting (5.19) into (5.17), one obtains the explicit expression for the mobility tensor. To make a comparison with Saffman's result, we will focus on the x,z-component of the mobility. The explicit expression is then given by

$$\mu_{xz}(0) = -\frac{1}{8\pi^{3/2}a\eta Re_\beta^{1/2}}\int\frac{\mathrm{d}\hat{\boldsymbol{k}}}{4\pi}\int_0^\infty \mathrm{d}t\,\frac{\hat{k}_x\hat{k}_z+\hat{k}_z^2 t}{c(\hat{\boldsymbol{k}},t)D^{1/2}(\hat{\boldsymbol{k}},t)}\exp\left[-\frac{Re_\beta\left\{c(\hat{\boldsymbol{k}},t)-1\right\}^2}{4D(\hat{\boldsymbol{k}},t)}\right]$$
$$\times\left(1-\exp\left[-\frac{Re_\beta c(\hat{\boldsymbol{k}},t)}{D(\hat{\boldsymbol{k}},t)}\right]\right). \tag{5.24}$$

Up to the lowest order in $Re_\beta^{1/2}$, (5.24) can be rewritten as

$$\mu_{xz}(0) = -\frac{Re_\beta^{1/2}}{8\pi^{3/2}a\eta}\int\frac{\mathrm{d}\hat{\boldsymbol{k}}}{4\pi}\int_0^\infty \mathrm{d}t\,\frac{\hat{k}_z^2(1-2\hat{k}_x^2-\hat{k}_x\hat{k}_z t)}{\left\{t\left(1+\hat{k}_x\hat{k}_z t+\hat{k}_z^2 t^2/3\right)^3\right\}^{1/2}}. \tag{5.25}$$

The derivation of this equation is given in Appendix B. This integral is exactly the same as the one Saffman derived. By integrating numerically, one obtains

$$\mu_{xz}(0) = -\frac{1}{6\pi a\eta}\times 0.343 Re_\beta^{1/2}. \tag{5.26}$$

Likewise, it is possible to evaluate other components:

$$\boldsymbol{\mu}(0) = \frac{1}{6\pi a\eta}\left\{\boldsymbol{1} - \begin{pmatrix} 0.327 & 0 & 0.343 \\ 0 & 0.577 & 0 \\ 0.944 & 0 & 0.0735 \end{pmatrix} Re_\beta^{1/2}\right\}. \tag{5.27}$$

The details of the derivations are given in Appendix B.

This result shows that, besides the x,z-component which Saffman (1965) and McLaughlin focused on, the z,x-component and the diagonal components of the mobility tensor are also modified in the lowest order in $Re_\beta^{1/2}$. Our value for the x,z-component is identical to the value given by Saffman. Even though our analytical expressions for all components, (5.24) and (B 7)–(B 10), are the same as the ones given by Harper & Chang (1968), their numerical values for the y,y-, z,x- and z,z-components differ, in particular for the z,z-component considerably. The differences in the numerical values clearly have their origin in the numerical integration procedure, which is rather delicate for these terms.

One may also give the friction tensor, which is the inverse tensor of the mobility,

$$\xi(0) = \mu^{-1}(0) = 6\pi a\eta \left\{ 1 + \begin{pmatrix} 0.327 & 0 & 0.343 \\ 0 & 0.577 & 0 \\ 0.944 & 0 & 0.0735 \end{pmatrix} Re_\beta^{1/2} \right\} \tag{5.28}$$

to the lowest order in $Re_\beta^{1/2}$.

6. Discussion

We have applied the induced force method to calculate the mobility tensor for a sphere moving relative to a fluid in arbitrary homogeneous flow. In this derivation, we used a general solution, constructed in §3, for the velocity field valid for an arbitrary choice of this homogeneous flow. In this solution, the velocity field is written as the sum of the incident homogeneous flow field and the velocity field due to the (induced) force distribution on the surface of the sphere. The nature of the second term in this solution, (3.12), is very different from the analogous term in the solution of the Stokes equation in which the inertial terms are neglected. In (3.12), the position of the induced force is replaced by the time-dependent position. This replacement corresponds to the transformation to the co-moving frame which is moving along with the local velocity field. By analysing the general solution, the closed form of the mobility tensor which is valid even for the time-dependent motion of the sphere in arbitrary homogeneous flow was derived up to the first order in $Re_\beta^{1/2}$. As a special case, we calculated the mobility for the short-time regime (the large-frequency regime) as well as the stationary case (the small-frequency limit) for simple shear flow. The value for the x,z-component which we find for the stationary case agrees with Saffman's result (Saffman 1965). This component gives a lift force on the sphere if it lags behind relative to the shear velocity. Saffman derived this lift force using the method of matched asymptotic expansions. The analysis by Harper & Chang was a straightforward extension of Saffman's analysis to the other components. Our analytical results for the sationary case agree with the expressions given by Harper *et al.*, but our numerical for the integrals values differ, often rather considerably, from theirs.

Equation (5.28) for the friction tensor shows that most of the components are modified in the same manner as the x,z-component. In particular, the z,x-component leads to another type of lift force. If one considers a particle suspended in Poiseuille flow, the x,z-component of the friction tensor causes the particle to be accelerated toward the axis if the particle lags behind relative to the Poiseuille flow. This was pointed out by Saffman (1965). If the particle moves toward the centre of the tube, however, the z,x-component of the friction tensor causes the particle to be accelerated in the direction opposite to the flow. We use the Poiseuille flow in this context only to illustrate the consequences of both off-diagonal terms in the friction tensor. Poiseuille flow is certainly not a convenient environment to measure these terms as other contributions which in fact are more important occur (Ishii & Hasimoto 1980). We are not aware of any experimental results which confirm the values of the x, z- and z, x-components of the friction tensor.

Our analysis presented here was restricted to the case of the same condition for the Reynolds numbers as Saffman's, $Re_u \ll Re_\beta^{1/2} \ll 1$. This can be extended in a straightforward manner to the regime which McLaughlin explored, $Re_u, Re_\beta^{1/2} \ll 1$. In that case, the Oseen term in (2.9) cannot be neglected. This results in the fact that the matrix $\mathbf{g}(\boldsymbol{k})$ in (3.4) has to be replaced by

$$\mathbf{g}(\boldsymbol{k}, t) = \nu k^2 - \mathrm{i}\boldsymbol{k} \cdot \Delta \boldsymbol{u}(t) + (\mathbf{1} - 2\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}) \cdot \boldsymbol{\beta}. \tag{6.1}$$

One can then perform the analysis, in the same way as presented in this paper, only for the stationary case, i.e. $\Delta \boldsymbol{u} = constant$. In this case, the expression for the mobility tensor reduces to McLaughlin's one. Extension to the non-stationary case is not trivial. McLaughlin (1993) also considered the case where the sphere is moving close to a wall. The induced force method is capable of dealing with this case, too, but we do not give a detailed analysis here as it is outside the scope of this paper.

We have analysed only the case of simple shear. Another example, pure rotational flow where $\boldsymbol{\beta} = -\boldsymbol{\beta}^\dagger$, was given by one of the authors using a formula which he derived (Miyazaki 1995) and it reproduced the values which have been derived by the method of matched asymptotic expansions. It seems clear that the induced force method is a simple and powerful alternative to the method of matched asymptotic expansions. This conclusion is further supported by the work of Weisenborn (1985) who analysed the motion of a sphere moving along the axis in a rotating cylinder using the same method. In both cases, the method gave rise to results which generalized the results found using the method of matched asymptotic expansions.

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Appendix A

In this appendix, we will give the derivation of (3.3). As the velocity field in the absence of the sphere, $\boldsymbol{v}_0(\boldsymbol{k}, t)$, satisfies (3.2) without the induced force field, $\boldsymbol{F}_{ind}(\boldsymbol{k}, t)$, the equation for the perturbation due to the presence of the sphere, $\delta\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v}_0$, is given by

$$\left.\begin{aligned} &\rho\left(\frac{\partial}{\partial t} + \nu k^2 + \boldsymbol{\beta} - \boldsymbol{k} \cdot \boldsymbol{\beta} \cdot \frac{\partial}{\partial \boldsymbol{k}}\right) \cdot \delta\boldsymbol{v}(\boldsymbol{k}, t) = -\mathrm{i}\boldsymbol{k}(p - p_0) + \boldsymbol{F}_{ind}(\boldsymbol{k}, t), \\ &\boldsymbol{k} \cdot \delta\boldsymbol{v} = 0, \end{aligned}\right\} \tag{A 1}$$

where p_0 is the hydrostatic pressure in the absence of the sphere. Operating with $\mathbf{P}_k = \mathbf{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}$ on both sides of (A 1) and using the incompressible nature of the fluid, one obtains

$$\left\{\frac{\partial}{\partial t}+\nu k^2+(\mathbf{1}-2\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot\boldsymbol{\beta}-\boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial \boldsymbol{k}}\right\}\cdot\delta\boldsymbol{v}(\boldsymbol{k},t)=\frac{1}{\rho}\boldsymbol{P}_{\boldsymbol{k}}\cdot\boldsymbol{F}_{ind}(\boldsymbol{k},t), \tag{A 2}$$

where the following relation has been used:

$$\begin{aligned}(\mathbf{1}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot\left(\boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)\delta\boldsymbol{v}&=\left(\boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)(\mathbf{1}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot\delta\boldsymbol{v}-\left\{\left(\boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial\boldsymbol{k}}\right)(\mathbf{1}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\right\}\cdot\delta\boldsymbol{v}\\&=\boldsymbol{k}\cdot\boldsymbol{\beta}\cdot\frac{\partial}{\partial\boldsymbol{k}}\delta\boldsymbol{v}+\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\cdot\boldsymbol{\beta}\cdot\delta\boldsymbol{v}.\end{aligned} \tag{A 3}$$

This together with the solution in the absence of the sphere leads to the formal solution, (3.3).

Appendix B

In this appendix, we will give the derivation of (5.25) more in detail. In (5.24), the integration over t can be divided into two parts as follows:

$$\mu_{xz}(0)=-\frac{1}{8\pi a\eta\epsilon^{1/2}}\int\frac{\mathrm{d}\hat{\boldsymbol{k}}}{4\pi}\left(\int_0^{\delta}\mathrm{d}t+\int_{\delta}^{\infty}\mathrm{d}t\right)f(\hat{\boldsymbol{k}},t), \tag{B 1}$$

where $f(\hat{\boldsymbol{k}},t)$ is the integrand of (5.24), $\epsilon\equiv Re_{\beta}$ and δ is chosen such that

$$\epsilon\ll\delta\ll 1. \tag{B 2}$$

One may expand each of the integrands in (B 1) in terms of appropriate parameters. For the first integral, as t is much smaller than 1, one may expand in powers of t

$$f(\hat{\boldsymbol{k}},t)=\frac{1}{\pi^{1/2}t^{1/2}}\left(1-\mathrm{e}^{-\epsilon/t}\right)\left[\hat{k}_x\hat{k}_z+\hat{k}_z^2\left(1-\frac{3}{2}\hat{k}_x^2\right)t\right]+O(t^2). \tag{B 3}$$

Integrating over $0\leqslant t\leqslant\delta$, one obtains up to the first order in $\delta^{1/2}$

$$\frac{1}{\epsilon^{1/2}}\int_0^{\delta}\mathrm{d}t\,f(\hat{\boldsymbol{k}},t)=\hat{k}_x\hat{k}_z\left\{2-\frac{2}{\pi^{1/2}}\left(\frac{\epsilon}{\delta}\right)^{1/2}\right\}+\frac{1}{\pi^{1/2}}\hat{k}_z^2\left(2-3\hat{k}_x^2\right)(\epsilon\delta)^{1/2}+O(\delta). \tag{B 4}$$

Likewise, in the outer region $\delta\leqslant t\leqslant\infty$, one can expand in powers of ϵ:

$$f(\hat{\boldsymbol{k}},t)=\frac{\epsilon(\hat{k}_x\hat{k}_z-\hat{k}_z^2t)}{\pi^{1/2}D^{3/2}(\hat{\boldsymbol{k}},t)}+O(\epsilon^2). \tag{B 5}$$

Using $\mathrm{d}D(\hat{\boldsymbol{k}},t)/\mathrm{d}t=c^2(\hat{\boldsymbol{k}},t)$,

$$\begin{aligned}\frac{1}{\epsilon^{1/2}}\int_{\delta}^{\infty}\mathrm{d}t\,f(\hat{\boldsymbol{k}},t)&\approx\frac{\epsilon^{1/2}}{\pi^{1/2}}\int_{\delta}^{\infty}\mathrm{d}t\left[\frac{\hat{k}_x\hat{k}_zc^2(\hat{\boldsymbol{k}},t)}{D^{3/2}(\hat{\boldsymbol{k}},t)}-\frac{\hat{k}_x\hat{k}_z\left\{c^2(\hat{\boldsymbol{k}},t)-1\right\}}{D^{3/2}(\hat{\boldsymbol{k}},t)}+\frac{\hat{k}_z^2t}{D^{3/2}(\hat{\boldsymbol{k}},t)}\right]\\&=\frac{2\hat{k}_x\hat{k}_z}{\pi^{1/2}}\left(\frac{\epsilon}{\delta}\right)^{1/2}-\frac{\hat{k}_z^2(2-3\hat{k}_x^2)}{\pi^{1/2}}(\epsilon\delta)^{1/2}+\frac{\epsilon^{1/2}}{\pi^{1/2}}\int_0^{\infty}\mathrm{d}t\,\frac{\hat{k}_z^2t\left(1-2\hat{k}_x^2-\hat{k}_x\hat{k}_zt\right)}{D^{3/2}(\hat{\boldsymbol{k}},t)}\\&\quad+O(\epsilon/\delta).\end{aligned} \tag{B 6}$$

Combining (B 4) and (B 6), one finds that the first two terms on the right-hand side of (B 6) cancel the corresponding terms in (B 4). Neglecting the higher-order terms in ϵ, one obtains (5.25).

The approximate expressions for other components are also evaluated in the same way. We will only list the results. After lengthy manipulation, one obtains

$$\mu_{xx}(0) = \frac{1}{6\pi a\eta}\left\{1 - \frac{3\epsilon^{1/2}}{16\pi^{3/2}}\int \mathrm{d}\hat{\boldsymbol{k}}\int_0^\infty \mathrm{d}t\,\Delta_{xx}(\hat{\boldsymbol{k}},t)\right\}, \tag{B 7}$$

$$\mu_{yy}(0) = \frac{1}{6\pi a\eta}\left\{1 - \frac{3\epsilon^{1/2}}{16\pi^{3/2}}\int \mathrm{d}\hat{\boldsymbol{k}}\int_0^\infty \mathrm{d}t\,\Delta_{yy}(\hat{\boldsymbol{k}},t)\right\}, \tag{B 8}$$

$$\mu_{zx}(0) = -\frac{1}{6\pi a\eta}\frac{3\epsilon^{1/2}}{16\pi^{3/2}}\int \mathrm{d}\hat{\boldsymbol{k}}\int_0^\infty \mathrm{d}t\,\Delta_{zx}(\hat{\boldsymbol{k}},t), \tag{B 9}$$

$$\mu_{zz}(0) = \frac{1}{6\pi a\eta}\left\{1 - \frac{3\epsilon^{1/2}}{16\pi^{3/2}}\int \mathrm{d}\hat{\boldsymbol{k}}\int_0^\infty \mathrm{d}t\,\Delta_{zz}(\hat{\boldsymbol{k}},t)\right\}, \tag{B 10}$$

where

$$\Delta_{xx}(\hat{\boldsymbol{k}},t) = \frac{\left(2\hat{k}_x + \hat{k}_z t\right)\left(1-\hat{k}_x^2\right)\hat{k}_z t}{D^{3/2}(\hat{\boldsymbol{k}},t)}, \tag{B 11}$$

$$\Delta_{yy}(\hat{\boldsymbol{k}},t) = \frac{\hat{k}_z}{D^{3/2}(\hat{\boldsymbol{k}},t)}\left[(2\hat{k}_x+\hat{k}_z t)(1-\hat{k}_y^2)t - \frac{\hat{k}_x\hat{k}_y^2 t}{1-\hat{k}_x^2} + \frac{(\hat{k}_x+\hat{k}_z t)\hat{k}_y^2}{1-\hat{k}_x^2}I(\hat{\boldsymbol{k}},t)\right], \tag{B 12}$$

$$\Delta_{zx}(\hat{\boldsymbol{k}},t) = \frac{1}{D^{3/2}(\hat{\boldsymbol{k}},t)}\left[-(2\hat{k}_x+\hat{k}_z t)\hat{k}_x\hat{k}_z^2 t + \hat{k}_y^2 I(\hat{\boldsymbol{k}},t)\right], \tag{B 13}$$

$$\Delta_{zz}(\hat{\boldsymbol{k}},t) = \frac{\hat{k}_z}{D^{3/2}(\hat{\boldsymbol{k}},t)}\left[(2\hat{k}_x+\hat{k}_z t)(1-\hat{k}_z^2)t - \frac{\hat{k}_x\hat{k}_z^2 t}{1-\hat{k}_x^2} - \frac{(\hat{k}_x+\hat{k}_z t)\hat{k}_y^2}{1-\hat{k}_x^2}I(\hat{\boldsymbol{k}},t)\right]. \tag{B 14}$$

The other components are found to be zero after integrating over $\hat{\boldsymbol{k}}$. Numerical integration of the above expressions leads to the results given in (5.27).

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